SMOOTHLY COMPACTIFIABLE SHEAR-FREE HYPERBOLOIDAL DATA IS DENSE IN THE PHYSICAL TOPOLOGY

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Abstract. We show that any polyhomogeneous asymptotically hyperbolic constant-mean-curvature solution to the vacuum Einstein constraint equations can be approximated, arbitrarily closely in Hölder norms determined by the physical metric, by shear-free smoothly conformally compact vacuum initial data.

Introduction

In the study of asymptotically flat (or asymptotically simple) spacetimes, initial data corresponding to spacelike slices extending towards null infinity has asymptotically hyperbolic geometry. Lars Andersson and Piotr Chruściel, building on their work with Helmut Friedrich [5], construct in [4] a large number of constant-mean-curvature (CMC) vacuum initial data sets with asymptotically hyperbolic geometry using the conformal method of Yvonne Choquet-Bruhat, André Lichnerowicz, and James York. In the work [4], particular attention is paid to the regularity of solutions at the conformal boundary. Data constructed in [4] typically admits a C^2, but not C^3 conformal compactification. In particular, they showed that data which is smooth in the interior "physical" manifold is typically polyhomogeneous, rather than smooth, at the conformal boundary.

In their detailed analysis [3], Andersson and Chruściel show that initial data must satisfy the shear-free condition along the conformal boundary (see §1.2) in order for any resulting spacetime geometry to admit a C^2 conformal compactification. This suggests that one might require the shear-free condition hold in order for a solution to the Einstein constraint equations to be "admissible" in the asymptotically hyperbolic setting. Thus we refer to initial data satisfying the shear-free condition as ***hyperboloidal***, distinguished among those solutions to the constraint equations having asymptotically hyperbolic geometry. Our recent work [1], joint with James Isenberg and John M. Lee, contains a systematic study of CMC hyperboloidal initial data, including a parametrization of all such data in the "weakly asymptotically hyperbolic" setting (see also [2]).

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This is not, however, the end of the story. Even if one restricts attention to shear-free data, the initial data constructed in [4] and [1] may not be sufficiently regular at the conformal boundary to obtain a spacetime development admitting conformal compactification. For example, the existing evolution theorems of Helmut Friedrich [8], [9], etc., all require more regularity of the conformal compactification. (The regularity issue is not unrelated to the shear-free condition: Andersson, Chruściel, and Friedrich show in [5] that initial data, constructed from smooth "free data" using the conformal method (see §1.3), with pure-trace extrinsic curvature is shear-free if and only if it is smoothly conformally compact.)

In addition to issues of regularity, one may be concerned about whether the collection of hyperboloidal data is sufficiently general for modeling a wide variety of physical situations.

Here we address these issues by showing that any polyhomogeneous asymptotically hyperbolic CMC solution to vacuum constraint equations can be approximated, arbitrarily closely in Hölder norms determined by the physical (non-compactified) spatial metric, by hyperboloidal (i.e. shear-free) vacuum initial data that is smoothly conformally compact. In the case that the conformal boundary is a 2-sphere, the work of [8] implies that the approximating data has a spacetime development admitting a smooth conformal infinity.

There are a number of ways in which one might interpret our result. From the perspective of modeling isolated gravitational systems, it is an indication that some version of Bondi-Sachs-Penrose approach to using conformal compactness for studying asymptotically flat spacetimes is feasible for studying a large class of physical systems. However, it is also an indication that the Hölder topology determined by the physical metric is insufficiently strong for studying the conformal boundary of asymptotically hyperbolic initial data sets. (For example, it is observed in [3] that among the initial data constructed in [4] from smooth "free data" by means of the conformal method, the shear-free condition does not hold generically with respect to the C^∞ topology determined by the conformally compactified metric.) Indeed, it was the approximation result here that motivated several of the results in [1], where continuity of the conformal method for construction of solutions to the constraint equations is established in a topology strong enough to detect the shear-free condition.

1. Discussion of main result

Here we present a discussion of the details needed in order to make precise our approximation result. As we make use of several results from [11], [2], and [1], we maintain conventions similar to the conventions in those works.

1.1. **Asymptotically hyperbolic initial data.** Let M be the interior of a smooth three-dimensional compact manifold $\overline{M}$ having boundary ∂M. We say that a smooth function $\rho\colon \overline{M} \to [0,\infty)$ is a ***defining function*** if $\rho^{-1}(0) = \partial M$ and if $d\rho \neq 0$ on ∂M. A metric g on M is said to be $\boldsymbol{C^k}$ ***conformally compact*** if $\overline{g} := \rho^2 g$ extends to a metric of class C^k on $\overline{M}$ for one, and hence all, smooth defining functions ρ. A C^2 conformally compact metric g is ***asymptotically hyperbolic*** if $|d\rho|_{\overline{g}} = 1$ along ∂M for one, and hence all, smooth defining functions ρ. The sectional curvatures of such metrics approach -1 as $\rho \to 0$; see [2] for generalizations of this definition.

A vacuum initial data set (g,K) consists of a Riemannian metric g and symmetric covariant 2-tensor K, both defined on M and satisfying the vacuum Einstein constraint equations

$$R[g] - |K|_g^2 + (\operatorname{tr}_g K)^2 = 0, \tag{1.1a}$$

$$\operatorname{div}_g K - d(\operatorname{tr}_g K) = 0. \tag{1.1b}$$

It is convenient to introduce the notation $\tau = \operatorname{tr}_g K$ for the trace of K and $\Sigma = K - \frac{1}{3}\tau g$ for the traceless part of K. We say that $\boldsymbol{(g,K)}$ ***is an asymptotically hyperbolic initial data set*** if g is asymptotically hyperbolic and if the tensor $\overline{\Sigma} = \rho\Sigma$ extends to a C^1 tensor field on $\overline{M}$. Such a data set is said to be ***smoothly conformally compact*** if for any defining function ρ the tensor fields $\overline{g} = \rho^2 g$ and $\overline{\Sigma} = \rho\Sigma$ extend smoothly to $\overline{M}$. We note that there exist "weakly asymptotically hyperbolic" solutions to (1.1), satisfying less stringent regularity conditions; see [1].

Asymptotically hyperbolic data sets may be viewed as intersecting future null infinity in the asymptotically flat spacetime containing a future development of the data set; we refer the reader to [1], and the references therein, for a more detailed discussion of asymptotically hyperbolic initial data sets and asymptotic flatness.

The formula for the change of scalar curvature under conformal deformation, together with (1.1a), implies

$$4\rho\Delta_{\overline{g}}\rho + (R[\overline{g}] - |\overline{\Sigma}|_{\overline{g}}^2)\rho^2 + 6\left(\frac{\tau^2}{9} - |d\rho|_{\overline{g}}^2\right) = 0, \tag{1.2}$$

where our sign convention on the scalar Laplace operator is $\Delta_{\overline{g}} = \operatorname{tr}_{\overline{g}} \operatorname{Hess}_{\overline{g}}$. Evaluating (1.2) at $\rho = 0$ we find that $\tau^2 = 9$ along ∂M. Thus in the constant-mean-curvature (CMC) setting we have $\tau = \pm 3$, with the sign indicating whether the initial data intersects future or past null infinity (relative to the notion of "future" determined by K). Henceforth we restrict attention to the CMC case and set $\tau = -3$, which (due to our sign convention for K) corresponds to future null

infinity; see the discussion in [1]. Note that when $\tau = -3$ the constraint equations (1.1) reduce to

$$R[g] - |\Sigma|_g^2 + 6 = 0 \quad \text{and} \quad \operatorname{div}_g \Sigma = 0. \tag{1.3}$$

1.2. **The shear-free condition.** While any sufficiently regular solution to the Einstein constraint equations (1.1) gives rise to some spacetime development thereof (see [6] and the references therein), it was shown in [3] that the development of an asymptotically hyperbolic initial data set admits a conformal compactification along future null infinity only if the ***shear-free condition***

$$\left[\operatorname{Hess}_{\overline{g}} \rho - \frac{1}{3}(\Delta_{\overline{g}}\rho)\overline{g} - \overline{\Sigma}\right]_{\partial M} = 0 \tag{1.4}$$

holds. We say that an asymptotically hyperbolic initial data set is a ***hyperboloidal initial data set*** if (1.4) holds.

1.3. **The conformal method.** The existence of asymptotically hyperbolic initial data sets is addressed in [5] and [4]. The existence of hyperboloidal data is discussed in [1]. All these works make use of the conformal method, which we now describe.

We first introduce the ***conformal Killing operator*** $\mathcal{D}_g$, which maps vector fields to trace-free symmetric covariant 2-tensors by

$$\mathcal{D}_g W = \frac{1}{2}\mathcal{L}_W g - \frac{1}{3}(\operatorname{div}_g W)g. \tag{1.5}$$

The formal L^2 adjoint $\mathcal{D}_g^*$ is given by $\mathcal{D}_g^* T = -(\operatorname{div}_g T)^\sharp$, and can be used to construct the self-adoint, elliptic operator $L_g := \mathcal{D}_g^*\mathcal{D}_g$, which is called the ***vector Laplacian***.

In the CMC setting, with $\tau = -3$, the conformal method seeks a solution (g, K) to (1.1) of the form

$$g = \phi^4\lambda \tag{1.6a}$$

$$K = \phi^{-2}\left(\mu + \mathcal{D}_\lambda W\right) - \phi^4\lambda, \tag{1.6b}$$

for some Riemannian metric λ, symmetric covariant 2-tensor field μ, vector field W, and positive function ϕ. Replacing g and K in (1.1) by the expressions in (1.6), we find that the constraints (1.1) are satisfied if W and ϕ satisfy the elliptic system

$$L_\lambda W = -\operatorname{div}_\lambda \mu \tag{1.7a}$$

$$\Delta_\lambda\phi = \frac{1}{8}\,\mathrm{R}[\lambda]\phi - \frac{1}{8}|\mu + \mathcal{D}_\lambda W|_\lambda^2\phi^{-7} + \frac{3}{4}\phi^5. \tag{1.7b}$$

Thus if λ and μ are specified, it remains only to solve (1.7) in order to obtain a solution to (1.1). We make use of the nomenclature of [1] and refer to (λ, μ) as a ***free data set***.

If λ is an asymptotically hyperbolic metric on M, then $g = \phi^4\lambda$ is an asymptotically hyperbolic metric provided $\phi \in C^2(\overline{M})$ and $\phi = 1$ along ∂M. If ϕ satisfies these conditions, then metric g and tensor K given by (1.6) form an asymptotically hyperbolic data set provided $\rho\mu$ and $\rho\mathcal{D}_\lambda W$ extend to C^1 tensor fields on $\overline{M}$.

We remark that the conformal method, as described above, does not necessarily yield hyperboidal data (i.e., data satisfying the shear-free condition). However, with appropriately constructed free data, one can ensure that the resulting initial data does in fact satisfy the shear-free condition; see [1].

1.4. **Polyhomogeneous data.** The two works [5] and [4], where large classes of asymptotically hyperbolic initial data are constructed, contain detailed analyses of the regularity of solutions at ∂M and show the following: Even if the free data λ and μ are smoothly conformally compact, the solutions W and ϕ to (1.7) need not give rise to smoothly conformally compactifiable fields g and K. Rather, the resulting metric g and tensor field K admit formal expansions at ∂M given, in terms of an arbitrary smooth defining function ρ, by

$$\overline{g} \sim \overline{g}_0 + \sum_{i=0}^{\infty}\sum_{p=0}^{p_i} \rho^{s_i}(\log\rho)^p \overline{g}_{ip}, \tag{1.8a}$$

$$\overline{\Sigma} \sim \overline{\Sigma}_0 + \sum_{i=0}^{\infty}\sum_{q=0}^{q_i} \rho^{t_i}(\log\rho)^q \overline{\Sigma}_{iq}, \tag{1.8b}$$

where the barred terms are smooth tensor fields. Tensor fields which admit such expansions are called "polyhomogeneous." We remark that a number of closely-related definitions of polyhomogeneous tensor fields exist in the literature; see §3 below for a precise definition of the notion of polyhomogenity used here.

The asymptotic expansions of the polyhomogeneous data constructed in [4] take the form (1.8) with $\mathrm{Re}(s_0) > 2$ and $\mathrm{Re}(q_0) > 1$. Thus, letting $C^k_{\mathrm{phg}}(\overline{M})$ denote the collection of polyhomogeneous tensor fields on M which extend to fields of class C^k on $\overline{M}$, we have $\overline{g} \in C^2_{\mathrm{phg}}(\overline{M})$ and $\overline{\Sigma} \in C^1_{\mathrm{phg}}(\overline{M})$. The polyhomogeneous hyperboloidal data sets constructed in [1] also have this regularity.

1.5. **The approximation theorem.** We now give a careful statement of our main result.

Theorem 1. *Suppose that (g, K) is a polyhomogeneous asymptotically hyperbolic vacuum initial data set on 3-manifold M. Then there exists a family $(g_\varepsilon, K_\varepsilon)$ of solutions to the vacuum Einstein constraint equations* (1.1)*, defined for sufficiently small $\varepsilon > 0$, such that*

(a) each initial data set is hyperboloidal, meaning that each (M, g_ε) *is asymptotically hyperbolic, that* $(g_\varepsilon, K_\varepsilon)$ *each satisfy the constraint equations* (1.1), *and that* $(g_\varepsilon, K_\varepsilon)$ *each satisfy the shear-free condition* (1.4)*;*

(b) each initial data set in the family is smoothly conformally compact, in the sense that $\overline{g}_\varepsilon = \rho^2 g_\varepsilon \in C^\infty(\overline{M})$ *and* $\overline{\Sigma}_\varepsilon = \rho(K_\varepsilon + g_\varepsilon) \in C^\infty(\overline{M})$*; and*

(c) we have $(g_\varepsilon, K_\varepsilon) \to (g, K)$ *as* $\varepsilon \to 0$ *in the* $C^{k,\alpha}(M) \times C^{k,\alpha}(M)$ *topology, for any* k *and* α*.*

We now describe the proof of Theorem 1; the details are contained in §4–§5 below. First we construct a family of free data $(\lambda_\varepsilon, \mu_\varepsilon)$ for small $\varepsilon > 0$. Our construction is such that the metrics λ_ε agree with g away from a neighborhood of ∂M, but are smoothly conformally compact. We also arrange that the fields μ_ε agree with Σ away from a neighborhood of ∂M, but are deformed near the boundary in order that the shear-free condition holds upon deformation to a solution of the constraint equations. The free data $(\lambda_\varepsilon, \mu_\varepsilon)$ is furthermore carefully constructed so that application of the conformal method yields smoothly conformally compact initial data sets. (The construction is motivated by the analysis in [3].) The proof proceeds by applying the conformal method to the free data $(\lambda_\varepsilon, \mu_\varepsilon)$. In order to show that the resulting solutions to the constraint equations approach (g, K) as $\varepsilon \to 0$, it is necessary to obtain uniform estimates for family of solutions W_ε, ϕ_ε to (1.7).

2. Technical preliminaries

We present several technical results needed for the proof of Theorem 1.

2.1. **Function spaces.** We fix a smooth defining function ρ on $\overline{M}$, and we make use of weighted Hölder spaces $C^{k,\alpha}_\delta(M)$ of tensor fields on M as defined in [2] (see also [11]). These spaces are defined independently of any Riemannian structure, but have equivalent norms determined by any sufficiently regular asymptotically hyperbolic metric. We emphasize that the convention regarding the weight δ is such that tensor field $u \in C^0_\delta(M)$ when $|u|_g \leq C\rho^\delta$ for any asymptotically hyperbolic metric $g = \rho^{-2}\overline{g}$. Recall that the ***weight of a tensor bundle*** is the covariant rank less the contravariant rank. (Thus the weight of a vector field is -1, while the weight of a metric tensor is 2.) The weight of a tensor field is important to keep in mind: If u is a tensor field of weight r, then $|u|_g = \rho^{-r}|u|_{\overline{g}}$. In particular, for tensors of weight r we have the following inclusion:

$$C^{k,\alpha}(\overline{M}) \subseteq C^{k,\alpha}_r(M);$$

compare with Lemma 3.7 of [11].

It is convenient to distinguish the following class of metrics: We say that an asymptotically hyperbolic metric h is a ***preferred background metric*** if $\overline{h} = \rho^2 h$ extends smoothly to $\overline{M}$ and if in a neighborhood of ∂M we have that $\overline{h}$ is a product metric of the form $d\rho \otimes d\rho + \overline{b}$ for some metric $\overline{b}$ on ∂M. We denote by ∇ and $\overline{\nabla}$ the Levi-Civita connections associated to h and $\overline{h}$ respectively, and note that the difference tensor $\overline{\nabla} - \nabla$ is an element of $C^{k,\alpha}(M)$ for all k and α. Throughout this section and the next, h represents any preferred background metric, and $\overline{h} = \rho^2 h$. In §4 we fix a preferred background metric, adapted to the metric g appearing in Theorem 1, which we retain throughout the proof of that theorem.

The following is an immediate consequence of Lemma 2.2(d) of [2].

Lemma 2. *Suppose $u \in C^{k,\alpha}_r(M)$ is a tensor field of weight r such that $\overline{\nabla} u \in C^{k-1,\alpha}_{r+1}(M)$ and such that $|u|_{\overline{h}} \to 0$ as $\rho \to 0$. Then $u \in C^{k,\alpha}_{r+1}(M)$ and*

$$\|u\|_{C^{k,\alpha}_{r+1}(M)} \leq C \left(\|u\|_{C^{k,\alpha}_r(M)} + \|\overline{\nabla} u\|_{C^{k-1,\alpha}_{r+1}(M)} \right)$$

2.2. **Differential operators.** We now record several results concerning differential operators arising in the conformal method. A differential operator $\mathcal{P} = \mathcal{P}[g]$ of order l arising from a metric g is said to be ***geometric*** (in the sense of [11]) if in any coordinate frame the components of $\mathcal{P}u$ are linear functions of u and its derivatives, whose coefficients are universal polynomials in the components of g, their partial derivatives, and $\sqrt{\det g_{ij}}$, such that the coefficient of the jth derivative of u involves no more than $l - j$ derivatives of the metric. Such operators are *uniformly degenerate*; the mapping properties of such operators have been studied in [12], [11], [4]. Recently, in work [2] with James Isenberg and John M. Lee we have extended some of these results to the *weakly asympotically hyperbolic* setting. We recall here several results needed for the proof of Theorem 1; the aforementioned works apply in much more general settings.

The following proposition allows us to compare corresponding operators arising from different metrics.

Proposition 3 (Proposition 7.9 of [1])**.** *Let $k \geq 0$, $\alpha \in [0, 1)$, and $\delta \in \mathbb{R}$. Suppose $g \in C^{k,\alpha}(M)$ is an asymptotically hyperbolic metric, and that $\mathcal{P}$ is a geometric operator of order $l \leq k$. Then there exists $\varepsilon_* > 0$ and $C > 0$ such that for any asymptotically hyperbolic metric $g' \in C^{k,\alpha}(M)$ with $\|g - g'\|_{C^{k,\alpha}(M)} \leq \varepsilon_*$ we have*

$$\|\mathcal{P}[g]u - \mathcal{P}[g']u\|_{C^{k-l,\alpha}_\delta(M)} \leq C \|g - g'\|_{C^{k,\alpha}(M)} \|u\|_{C^{k,\alpha}_\delta(M)}$$

for all $u \in C^{k,\alpha}_\delta(M)$.

We now turn attention to elliptic geometric operators. The operators arising in the results here satisfy the following.

Assumption P. *Suppose (M,g) is an asymptotically hyperbolic manifold. We assume $\mathcal{P}=\mathcal{P}[g]$ is a second-order linear elliptic operator acting on sections of a tensor bundle E. Furthermore we assume that $\mathcal{P}$ is geometric in the sense defined above, and that $\mathcal{P}$ is formally self-adjoint.*

The mapping properties of operators satisfying Assumption P can be understood by studying the ***indicial map*** $I_s(\mathcal{P})$, defined for $s\in\mathbb{C}$ to be the bundle map

$$E\otimes\mathbb{C}\big|_{\partial M}\to E\otimes\mathbb{C}\big|_{\partial M}$$

given by $I_s(\mathcal{P})\overline{u}=\rho^{-s}\mathcal{P}(\rho^s\overline{u})\big|_{\rho=0}$; see [12], [11]. The ***characteristic exponents*** of $\mathcal{P}$, which we denote by $\mathcal{E}$, are defined to be those values of s for which $I_s(\mathcal{P})$ has a non-trivial kernel at some point on ∂M. In [11] it is shown that, under Assumption P, these exponents and their multiplicities are constant on ∂M, and agree with those associated to the corresponding operator in the half-space model of hyperbolic space. Furthermore, due to the self-adjointness of $\mathcal{P}$, the characteristic exponents are symmetric about the line $\mathrm{Re}(s)=1-r$, where r is the weight of E. The ***indicial radius*** R of $\mathcal{P}$ is defined to be the smallest number $R\geq 0$ such that $\mathrm{Re}(s)\leq 1-r+R$ for all $s\in\mathcal{E}$.

The importance of the indicial radius is the following result from [11]: If (M,g) is asymptotically hyperbolic of class $C^{k,\alpha}$ with $\alpha\in(0,1)$, if Assumption P is satisfied, and if there is a compact set $K\subset M$ and a constant $C>0$ such that

$$\|u\|_{L^2(M)}\leq C\|\mathcal{P}u\|_{L^2(M)} \quad \text{for all} \quad u\in C_c^\infty(M\smallsetminus K), \tag{2.1}$$

then $\mathcal{P}\colon C^{k+2,\alpha}_\delta(M)\to C^{k,\alpha}_\delta(M)$ is Fredholm if and only if $|1-\delta|<R$.

The following proposition is a consequence of [1, Proposition 6.3], [2, Proposition 6.1], and [2, Lemma 5.6] (see also [11, Lemma 6.4]); we emphasize that the results cited apply in much more general situations.

Proposition 4. *Suppose (M,g) is an asymptotically hyperbolic 3-manifold, and suppose that g is smoothly conformally compact.*

(a) For each $k\geq 0$, $\alpha\in(0,1)$, and $\delta\in(-1,3)$ the vector Laplacian is an isomorphism

$$L_g\colon C^{k+2,\alpha}_\delta(M)\to C^{k,\alpha}_\delta(M).$$

In particular, there exists a constant $C>0$ such that

$$\|X\|_{C^{k+2,\alpha}_\delta(M)}\leq C\|L_gX\|_{C^{k,\alpha}_\delta(M)}$$

for all vector fields $X\in C^{k+2,\alpha}_\delta(M)$.

(b) Let $k\geq 0$ and $\alpha\in(0,1)$. Suppose $\kappa\in C^{k,\alpha}_\sigma(M)$ for some $\sigma>0$ and that c is a constant satisfying $c>-1$ and $c+\kappa\geq 0$. Then so long as

$$|\delta-1|\leq\sqrt{1+c}$$

the map

$$\Delta_g - (c+\kappa)\colon C^{k+2,\alpha}_\delta(M) \to C^{k,\alpha}_\delta(M)$$

is an isomorphism. In particular, there exists a constant $C>0$ such that

$$\|u\|_{C^{k+2,\alpha}_\delta(M)} \le C\|\Delta_g u - (c+\kappa)u\|_{C^{k,\alpha}_\delta(M)}$$

for all functions $u \in C^{k+2,\alpha}_\delta(M)$.

Furthermore, if $w \in C^0_\delta(M)$ is such that $\Delta_g w - (c+\kappa)w \in C^{k,\alpha}_{\delta'}(M)$ then $w \in C^{k,\alpha}_{\delta'}(M)$ whenever $|\delta'-1| \le \sqrt{1+c}$.

2.3. **The tensor $\mathcal{H}_{\overline{g}}(\rho)$.** Together with James Isenberg and John M. Lee, we introduced in [2] a conformally invariant version of the trace-free Hessian that is used in [1] to characterize the shear-free condition; we now recall its definition and basic properties. Let

$$A_{\overline{g}}(\rho) = \frac{1}{2}|d\rho|_{\overline{g}}\operatorname{div}_{\overline{g}}\left[|d\rho|_{\overline{g}}\operatorname{grad}_{\overline{g}}\rho\right].$$

We define the tensor field $\mathcal{H}_{\overline{g}}(\rho)$ by

$$\mathcal{H}_{\overline{g}}(\rho) := |d\rho|^6_{\overline{g}}\,\mathcal{D}_{\overline{g}}(|d\rho|^{-2}_{\overline{g}}\operatorname{grad}_{\overline{g}}\rho) + A_{\overline{g}}(\rho)\left(d\rho\otimes d\rho - \frac{1}{3}|d\rho|^2_{\overline{g}}\,\overline{g}\right), \tag{2.2}$$

where $\mathcal{D}_{\overline{g}}$ is the conformal Killing operator defined in (1.5).

We have the following basic properties of $\mathcal{H}_{\overline{g}}(\rho)$.

Proposition 5 (Proposition 4.1 of [2])**.**

(a) $\mathcal{H}_{\overline{g}}(\rho)$ is symmetric and trace-free.
(b) $\mathcal{H}_{\overline{g}}(\rho)(\operatorname{grad}_{\overline{g}}\omega,\cdot) = 0$.
(c) $\mathcal{H}_{\overline{g}}(c\rho) = c^5\mathcal{H}_{\overline{g}}(\rho)$ for all constants c.
(d) If θ is a strictly positive function then $\mathcal{H}_{\theta^4\overline{g}}(\rho) = \theta^{-8}\mathcal{H}_{\overline{g}}(\rho)$ and $A_{\theta^4\overline{g}}(\rho) = \theta^{-8}A_{\overline{g}}(\rho)$.

Proposition 6 (Corollary 4.4 of [1])**.** *Suppose $(g,K) = (g,\Sigma - g)$ is a polyhomogeneous asymptotically-hyperbolic CMC initial data set. Then the shear-free condition* (1.4) *is satisfied if and only if*

$$\overline{\Sigma}\big|_{\partial M} = \mathcal{H}_{\overline{g}}(\rho)|_{\partial M}, \tag{2.3}$$

where $\overline{g} = \rho^2 g$ and $\overline{\Sigma} = \rho\Sigma$.

3. Analysis on $\overline{M}$

The solution to a geometric elliptic equation of the form $\mathcal{P}u = f$ on an asymptotically hyperbolic manifold (M,g) may be smooth on M, but may not extend smoothly to $\overline{M}$, even if $\overline{g}\in C^\infty(\overline{M})$; see [12], [4], [7], et. al. Rather, many solutions to elliptic equations have asymptotic expansions at ∂M containing powers of ρ and powers of $\log\rho$. The logarithmic terms arise in situations where there is a resonance

(see §3.3 or [2, Remark A.12]) and are thus features of the algebraic structure of $\mathcal{P}$. Tensor fields with expansions involving powers of ρ and $\log\rho$ are called *polyhomogeneous*. We now present a more careful definition, and subsequently discuss conditions under which the solution itself is in fact smooth on $\overline{M}$. We note that a number of related definitions of polyhomogeneity appear in the literature; see [12], [4], [2], [10], [7], et. al.

For convenience, we work with a fixed preferred background metric $h = \rho^{-2}\overline{h}$, denoting by $\overline{\nabla}$ the Levi-Civita connection associated to $\overline{h}$. (The following, however, is independent of the choice of h.) We subsequently make frequent and implicit use of the following construction: If E is a tensor bundle over $\overline{M}$ and $\overline{u}$ is a smooth section of $E\big|_{\partial M}$, we may extend $\overline{u}$ to the neighborhood of ∂M by parallel transport along $\operatorname{grad}_{\overline{h}}\rho$; using a smooth cutoff function, the resulting tensor may be extended further to all of $\overline{M}$. Furthermore, when working in the neighborhood of ∂M where $\overline{h} = d\rho\otimes d\rho + \overline{b}$, we abuse notation by writing $\rho\partial_\rho$ for $\rho\overline{\nabla}_{\operatorname{grad}_{\overline{h}}\rho}$.

3.1. **Polyhomogeneity.** In order to carefully define polyhomogeneity for tensor fields, we first introduce for each $\delta\in\mathbb{R}$ the class $\mathcal{B}_\delta(M)$ of tensor fields, defined by

$$\mathcal{B}_\delta(M) = \bigcap_{\substack{0\le k\\ t<\delta}} C^k_t(M).$$

(The reader may wish to compare these spaces to the conormality spaces appearing in [2] and the references therein.)

The importance of this definition is that if $s\in\mathbb{C}$ then $\rho^s\log\rho$ is contained in $\mathcal{B}_\delta(M)$ with $\delta = \operatorname{Re}(s)$, but is not of class $C^0_\delta(M)$. Furthermore, $u\in\mathcal{B}_\delta(M)$ if and only if $(\log\rho)u\in\mathcal{B}_\delta(M)$. If tensor u of weight r satisfies $u\in\mathcal{B}_{\delta+r}(M)$ for some $\delta>0$, then $(\rho\partial_\rho)^k u$ vanishes at ∂M for all $k\ge 0$. The same holds for certain other fields, such as the functions $\rho^s(\log\rho)^{-n}$ with $\operatorname{Re}(s)=0$ and n a positive integer. Consequently, we obtain the following.

Lemma 7. *Suppose E is a tensor bundle of weight r, and that $t_i\in\mathbb{C}$, $q_i\in\mathbb{N}_0$, and sections $\overline{u}_i$ of $E\big|_{\partial M}$ are such that*

$$u = \sum_{i=1}^N \rho^{t_i}(\log\rho)^{q_i}\overline{u}_i \in \mathcal{B}_{\delta+r}(M)$$

for some $\delta > \max_{1\le i\le N}\operatorname{Re}(t_i)$. Then $\overline{u}_i = 0$ for all $1\le i\le N$.

Proof. It suffices to fix a point of ∂M and consider the case when u is a function. Under such restrictions our claim is a consequence of the fact that a finite $\mathbb{R}$-linear combination of single-variable functions of the form $\rho^t(\log\rho)^q$ with $\operatorname{Re}(t)=0$

$$\sum_{j=1}^J a_j\rho^{ib_j}(\log\rho)^{q_j}$$

vanishes at $\rho = 0$ together with all of its $\rho\partial_\rho$ derivatives if and only if all of the coefficients a_j vanish. $\square$

A smooth section u of tensor bundle E over M having weight r is defined to be ***polyhomogeneous*** if

(a) there exist sequences $s_i \in \mathbb{C}$ and $p_i \in \mathbb{N}_0$ with $\mathrm{Re}(s_i)$ non-decreasing and diverging to $+\infty$ as $i \to \infty$,
(b) for $i, p \in \mathbb{N}_0$ with $0 \le p \le p_i$ there exists smooth section $\overline{u}_{ip}$ of $E\big|_{\partial M}$, and
(c) for each $k \in \mathbb{N}_0$ there exists $N_k \in \mathbb{N}_0$ such that

$$u - \sum_{i=0}^{N_k} \sum_{p=0}^{p_i} \rho^{s_i - r} (\log \rho)^p \overline{u}_{ip} \in \mathcal{B}_k(M). \tag{3.1}$$

We assume that those exponents s_i having the same real part are ordered such that their imaginary parts are increasing. (The factor of ρ^{-r} in (c) is motived by the fact that $|u|_g = \rho^r |u|_{\overline{g}}$; thus the leading order behavior of $|u|_g$ will be as $\rho^{\mathrm{Re}(s_0)}(\log\rho)^{p_0}$.)

If u satisfies the definition above, we write

$$u \sim \sum_{i=0}^{\infty} \sum_{p=0}^{p_i} \rho^{s_i - r} (\log \rho)^p \overline{u}_{ip}.$$

Let $\mathcal{B}_{\mathrm{phg}}(M)$ be the collection of all tensor fields on M which are polyhomogeneous as defined above. We furthermore denote by $\mathcal{B}^{\mathrm{phg}}_\delta(M)$ those polyhomogeneous tensor fields that are of class $\mathcal{B}_\delta(M)$, and by $C^k_{\mathrm{phg}}(\overline{M})$ those polyhomogeneous tensor fields extending to tensor fields of class C^k on $\overline{M}$.

Remark 8.

(a) It follows from Lemma 7 that if $u \in \mathcal{B}^{\mathrm{phg}}_\delta(M)$ then we have $\mathrm{Re}(s_0) \ge \delta$.
(b) Polyhomogeneous expansions are unique in the sense that if

$$v \sim \sum_{i=0}^{\infty} \sum_{p=0}^{p_i} \rho^{s_i} (\log \rho)^p \overline{v}_{ip} \quad \textit{and} \quad v \sim \sum_{i=0}^{\infty} \sum_{q=0}^{q_i} \rho^{t_i} (\log \rho)^p \overline{w}_{ip},$$

then $s_i = t_i$, $p_i = q_i$, and $\overline{v}_{ip} = \overline{w}_{jp}$.
(c) Tensor fields u which are smooth on $\overline{M}$ are polyhomogeneous with a Taylor-series like expansion

$$u \sim \sum_{n=0}^{\infty} \frac{\rho^n}{n!} \overline{u}_n.$$

The fields $\overline{u}_n$ are the restrictions of $\overline{\nabla}^n u(\mathrm{grad}_{\overline{h}}\,\rho, \dots, \mathrm{grad}_{\overline{h}}\,\rho, \cdot, \dots, \cdot)$ to the boundary. We emphasize that this holds regardless of whether u is analytic or not.
(d) A tensor field $u \in \mathcal{B}^{\mathrm{phg}}_\delta(M)$ of weight r is in $C^l_{\mathrm{phg}}(\overline{M})$ if $\delta > l + r$; see Lemma 3.7 in [11]. Thus $u \in C^\infty(\overline{M})$ if and only if $u \in C^\infty(\overline{M}) + \mathcal{B}^{\mathrm{phg}}_k(M)$

for all $k \in \mathbb{N}$. Furthermore, a polyhomogeneous tensor field

$$u \sim \sum_{i=0}^{\infty} \sum_{p=0}^{p_i} \rho^{s_i} (\log \rho)^p \overline{u}_{ip}.$$

is smooth on $\overline{M}$ if and only if $s_i \in \mathbb{N}_0$ and $p_i = 0$ for all i.

3.2. **PDE results.** The relationship between the uniformly degenerate elliptic operators and polyhomogeneity has been extensively studied in [12]; see also [4], [2], [1] for studies focusing on operators arising in the study of the Einstein constraint equations. It this paper we make use of the following result, which is a consequence of Proposition 6.3 of [1] and Proposition 6.4 of [2].

Proposition 9. *Suppose that (M, g) is a smoothly conformally compact asymptotically hyperbolic 3-manifold.*

(a) If Y is a vectorfield on M which extends smoothly to $\overline{M}$, then the solution W to

$$L_g W = Y$$

satisfies $W \in \rho^3 C^0_{\mathrm{phg}}(\overline{M})$ and $\mathcal{D}_g W \in C^0_{\mathrm{phg}}(\overline{M})$.

(b) For any function $A \in \rho^2 C^\infty(\overline{M})$, there exists a unique positive solution $\phi \in C^2_{\mathrm{phg}}(\overline{M})$ to

$$\Delta_g \phi = \frac{1}{8} R[g]\phi - A\phi^{-7} + \frac{3}{4}\phi^5, \qquad \phi|_{\partial M} = 1.$$

Furthermore, if $R[g] + 6 = \mathcal{O}(\rho^2)$ then $\phi - 1 = \mathcal{O}(\rho^2)$.

3.3. **Boundary regularity.** Even if g is smoothly conformally compact and f extends smoothly to $\overline{M}$, solutions to $\mathcal{P}[g]u = f$ may not extend smoothly to $\overline{M}$. To understand why this is the case, and to understand those circumstances where u *does* extend smoothly to $\overline{M}$, we examine more closely the relationship between $\mathcal{P}$ and its indicial map $I_s(\mathcal{P})$. For a more general treatment of the subject the reader is referred to [12]; see also [4],

In background coordinates $(\rho, \theta^1, \theta^2) = (\theta^0, \theta^1, \theta^2)$ near ∂M (see [2], [11]), we have

$$\mathcal{P} = a^{ij}(\rho\partial_i)(\rho\partial_j) + b^i(\rho\partial_i) + c,$$

where the matrix-valued functions a^{ij}, b^i, and c extend smoothly to $\rho = 0$. Computing in these coordinates one sees that

$$I_s(\mathcal{P})\overline{u} = \rho^{-s}\mathcal{P}(\rho^s \overline{u})\big|_{\rho=0} = (\overline{a}^{\rho\rho}s^2 + \overline{b}^\rho s + \overline{c})\overline{u},$$

where $\overline{a}^{\rho\rho} = a^{\rho\rho}\big|_{\rho=0}$, $\overline{b}^\rho = b^\rho\big|_{\rho=0}$, and $\overline{c} = c\big|_{\rho=0}$ are smooth (matrix-valued) functions of (θ^1, θ^2).

As in [12], we define the ***indicial operator*** $I(\mathcal{P})$ to be the unique dilation-invariant operator on $\partial M \times (0, \infty)$ such that

$$I(\mathcal{P})(\rho^s \overline{u}) = \rho^s I_s(\mathcal{P})\overline{u}$$

for all smooth sections $\overline{u}$ of $E\big|_{\partial M}$. Thus

$$I(\mathcal{P})(\rho^s (\log \rho)^p \overline{u}) = \sum_{k=0}^{p} \binom{p}{k} \rho^s (\log \rho)^{p-k} I_s^{(k)}(\mathcal{P})\overline{u}, \tag{3.2}$$

where $I_s^{(k)}(\mathcal{P}) = \frac{d^k}{ds^k} I_s(\mathcal{P})$. In coordinates we have

$$I(\mathcal{P}) = \overline{a}^{\rho\rho}(\rho\partial_\rho)^2 + \overline{b}^{\rho}(\rho\partial_\rho) + \overline{c},$$

with $\overline{a}^{\rho\rho}$, $\overline{b}^{\rho}$ and $\overline{c}$ as above. It should be noted that $I(\mathcal{P})$ can be extended to a differential operator $\mathcal{I}(\mathcal{P})u = I(\mathcal{P})(\varphi u)$ on M by means of a cut-off function φ supported in a collar neighborhood of ∂M. We furthermore set $\mathcal{R} = \mathcal{P} - \mathcal{I}(\mathcal{P})$.

Careful examinations of coordinate expressions for $\mathcal{P}$, $I(\mathcal{P})$ and $\mathcal{R}$ yield the following:

Lemma 10. *Suppose (M, g) is a smoothly conformally compact asymptotically hyperbolic manifold and that $\mathcal{P}$ satisfies Assumption P. Then for any $\delta \in \mathbb{R}$ we have*

(a) $\mathcal{I}(\mathcal{P})\colon \mathcal{B}_\delta^{\text{phg}}(M) \to \mathcal{B}_\delta^{\text{phg}}(M)$ and
(b) $\mathcal{R}\colon \mathcal{B}_\delta^{\text{phg}}(M) \to \mathcal{B}_{\delta+1}^{\text{phg}}(M)$.

This lemma can be interpreted as saying that $\mathcal{I}(\mathcal{P})$ is an approximation of $\mathcal{P}$ near ∂M. It is crucial to notice that $I(\mathcal{P})$ is an operator of Cauchy-Euler type. The method advertised in entry-level courses for solving a constant coefficient Cauchy-Euler ODE such as

$$\overline{a}(\rho\partial_\rho)^2 u + \overline{b}(\rho\partial_\rho)u + \overline{c}u = f \tag{3.3}$$

involves studying the roots s_1 and s_2 of the associated characteristic polynomial equation

$$\overline{a}s^2 + \overline{b}s + \overline{c} = 0.$$

In the PDE setting, this corresponds to a study of characteristic exponents as defined in §2.2.

Typical solutions to the ODE (3.3) have expansions in terms of powers of ρ, where the exponents present are the same as the exponents in the expansion of f, as well as the roots s_i. However, when the expansion of f includes ρ^{s_i}, we have a resonance that leads to the presence of terms of the form $\rho^{s_i} \log \rho$ in the expansion of the solution u. Further resonances arise when $s_1 = s_2$, in which case the two homogeneous solutions are ρ^{s_1} and $\rho^{s_1} \log \rho$.

The situation in the case of a (self-adjoint, geometric, elliptic) PDE in asymptotically hyperbolic setting is extremely similar to the ODE case. We now present conditions which ensure that no resonances, and thus no log terms, occur. The proofs presented below are inspired by computations done in [3].

Proposition 11. *Let (M, g) be an asymptotically hyperbolic manifold that is smoothly conformally compact. Suppose $\mathcal{P} = \mathcal{P}[g]$ acts on tensors of weight r and satisfies Assumption P, and let μ denote the maximum real part of the characteristic exponents of $\mathcal{P}$. If $u \in \mathcal{B}_{\mathrm{phg}}(M)$ is such that*

(a) $\mathcal{P}u$ extends to a tensor field in $C^\infty(\overline{M})$, and
(b) there exists $\delta > \mu$ such that $u \in C^\infty(\overline{M}) + \mathcal{B}^{\mathrm{phg}}_{\delta+r}(M)$,

then u extends to a tensor field in $C^\infty(\overline{M})$.

Proof. Without loss of generality we may assume $u \in \mathcal{B}^{\mathrm{phg}}_{\delta+r}(M)$. By Remark 8 we may then assume that u has polyhomogeneous expansion (3.1) with $\mathrm{Re}(s_i - r) > \mu$ for all i. Let $\{\delta_j\}_{j=0}^{\infty}$ be the strictly increasing sequence listing the elements of $\mathrm{Re}\{s_i\}$. It suffices to show that for each $j \in \mathbb{N}_0$ there exists $\overline{u}_j \in C^\infty(\overline{M})$ and $u_j \in \mathcal{B}^{\mathrm{phg}}_{\delta_j}(M)$ such that $u = \overline{u}_j + u_j$; we do so inductively.

When $j = 0$ there is nothing to prove as we may set $\overline{u}_0 = 0$. Thus we assume for some $j \geq 0$ that $u = \overline{u}_j + u_j$ as above. Let

$$w_j = \sum_{\mathrm{Re}(s_i)=\delta_j} \sum_{p=0}^{p_i} \rho^{s_i - r} (\log \rho)^p \overline{u}_{ip}$$

and define $u_{j+1} = u_j - w_j$; note that $u_{j+1} \in \mathcal{B}^{\mathrm{phg}}_{\delta_{j+1}}(M)$ as desired. From Lemma 10 we have that

$$\mathcal{I}(\mathcal{P}) w_j \in C^\infty(\overline{M}) + \mathcal{B}^{\mathrm{phg}}_{\delta'}(M), \quad \delta' > \delta_j. \tag{3.4}$$

On the other hand, a direct computation in a collar neighborhood of the boundary ∂M shows that

$$\mathcal{I}(\mathcal{P}) w_j = \sum_{\mathrm{Re}(s_i)=\delta_j} \sum_{p=0}^{p_i} \rho^{s_i - r} (\log \rho)^p \overline{w}_{ip},$$

where by (3.2) we have

$$\overline{w}_{ip_i} = I_{s_i - r}(\mathcal{P}) \overline{u}_{ip_i}, \quad \overline{w}_{i(p_i-1)} = I_{s_i-r}(\mathcal{P}) \overline{u}_{i(p_i-1)} + p_i I^{(1)}_{s_i-r}(\mathcal{P}) \overline{u}_{ip_i}, \tag{3.5}$$

etc.

In view of Remark 8 it follows that each exponent $s_i - r$ in the expansion of w_j is a non-negative integer and that $I_{s_i-r}(\mathcal{P})\overline{u}_{ip_i} = \overline{w}_{ip_i} = 0$ whenever $p_i \neq 0$. However, since $\mathrm{Re}(s_i - r) > \mu$ we can only have $I_{s_i-r}(\mathcal{P})\overline{u}_{ip_i} = 0$ if $\overline{u}_{ip_i} = 0$. Thus $p_i = 0$, and the proof of our induction step is complete. ∎

For simplicity, we now restrict attention to a special class of operators, which includes those arising in the conformal method.

Assumption L. *Suppose (M,g) is an asymptotically hyperbolic manifold, and that $\mathcal{P}=\mathcal{P}[g]$ is a geometric operator acting on sections of tensor bundle E and satisfying Assumption P. We furthermore assume that the indicial operator $I_s(\mathcal{P})$ is a product of a polynomial $p(s)$ and an isomorphism of $E\big|_{\partial M}$, where $p(s)$ has simple integer roots.*

Proposition 12. *Let (M,g) be an asymptotically hyperbolic manifold that is smoothly conformally compact. Suppose $\mathcal{P}=\mathcal{P}[g]$ acts on tensor field of weight r and satisfies Assumption L. Let μ denote the highest characteristic exponent of $\mathcal{P}$. If $u\in\mathcal{B}^{\mathrm{phg}}_{\mu+r}(M)$ satisfies $\mathcal{P}u\in C^\infty(\overline{M})$ and $\mathcal{P}u\in\mathcal{B}_{\delta+r}(M)$ for some $\delta>\mu\geq 0$, then u extends to a smooth tensor field on $\overline{M}$.*

Proof. Since $u\in\mathcal{B}^{\mathrm{phg}}_{\mu+r}(M)$, it admits an expansion (3.1) with $\mathrm{Re}(s_i-r)\geq\mu$. Let

$$w=\sum_{\mathrm{Re}(s_i-r)=\mu}\sum_{p=0}^{p_i}\rho^{s_i-r}(\log\rho)^p\overline{u}_{ip}. \tag{3.6}$$

From Lemma 10 we have

$$\mathcal{I}(\mathcal{P})w\in\mathcal{B}^{\mathrm{phg}}_{\delta'+r}(M)\ \text{ for some }\ \delta'>\mu.$$

The computation (3.5) and Remark 8 now imply that $I_{s_i-r}(\mathcal{P})\overline{u}_{ip_i}=0$ and, if $p_i\geq 1$, that

$$I_{s_i-r}(\mathcal{P})\overline{u}_{i(p_i-1)}+p_iI^{(1)}_{s_i-r}(\mathcal{P})\overline{u}_{ip_i}=0. \tag{3.7}$$

Therefore, the only non-vanishing term in the expansion (3.6) has to correspond to $s_i-r=\mu$ which, by our assumptions, is a nonnegative integer. Furthermore, we must have $p_i=0$ because otherwise (3.7) contradicts Assumption L. Thus w extends smoothly to $\overline{M}$ and our result is now immediate from Proposition 11. □

We conclude this section with a regularity result for semilinear scalar equations of the form $\mathcal{P}u=f(u)$, where f satisfies the following.

Assumption F. *We assume that f is a smooth real function on $M\times I$ where $0\in I$ is an open interval. Furthermore, we assume that on a neighborhood of zero f is given by an absolutely and uniformly convergent power series*

$$f(x,u)=\sum_{l=0}^{\infty}a_l(x)u^l$$

with functions $a_0,a_1\in\rho C^\infty(\overline{M})$ and $a_l\in C^\infty(\overline{M})$ for $l\geq 2$.

In what follows we simply write $f(u)$ for $f(\cdot,u(\cdot))$.

Remark 13. *If* $u \in \rho C^\infty(\overline{M}) + \mathcal{B}^{\mathrm{phg}}_\delta(M)$ *with* $\delta > 1$*, and if* f *satisfies Assumption F then* $f(u) \in \rho C^\infty(\overline{M}) + \mathcal{B}^{\mathrm{phg}}_{\delta+1}(M)$*.*

Proposition 14. *Let* (M, g) *be an asymptotically hyperbolic manifold that is smoothly conformally compact and suppose* $\mathcal{P} = \mathcal{P}[g]$ *is an elliptic operator acting on functions and satisfying Assumption L. Let* μ *denote the largest characteristic exponent of* $\mathcal{P}$*. Furthermore, let* f *be a function satisfying Assumption F.*

Suppose that $\mathcal{P}u = f(u)$*, where* $u \in \mathcal{B}^{\mathrm{phg}}_\mu(M)$ *and* $f(u) \in \mathcal{B}_\delta(M)$ *for some* $\delta > \mu$*. Then* u *extends to a function in* $C^\infty(\overline{M})$*.*

Proof. Since $u \in \mathcal{B}^{\mathrm{phg}}_\mu(M)$, it admits an expansion (3.1) with $\mathrm{Re}(s_i) \geq \mu$. Note that $\mu \geq 1$, as a consequence of the fact that the set of the characteristic exponents of $\mathcal{P}$ is symmetric about $\mathrm{Re}(s) = 1$ (cf. Corollary 4.5 in [11]). Furthermore, by Assumption L we have that μ is an integer.

As in the proof of Proposition 12 we consider the function

$$w = \sum_{\mathrm{Re}(s_i)=\mu} \sum_{p=0}^{p_i} \rho^{s_i} (\log \rho)^p \overline{u}_{ip}.$$

From Lemma 10 and the assumption that $f(u) \in \mathcal{B}_\delta(M)$ for some $\delta > \mu$ we have

$$I(\mathcal{P})w \in \mathcal{B}^{\mathrm{phg}}_{\delta'}(M) \quad \text{for some} \quad \delta' > \mu.$$

Arguing as in the proof of Proposition 12 we obtain $s_i = \mu$ and $p_i = 0$ for all i in the above expression for w. Thus

$$u = \rho^\mu \overline{u}_{\mu 0} + v \in \rho C^\infty(\overline{M}) + \mathcal{B}^{\mathrm{phg}}_{\delta''}(M) \quad \text{for some} \quad \delta'' > \mu.$$

It remains to establish smoothness of the function v. We do so by using the inductive argument from the proof of Proposition 11 to show that for each j there exist $\overline{v}_j \in \rho C^\infty(\overline{M})$ and $v_j \in \mathcal{B}^{\mathrm{phg}}_{\delta_j}(M)$ such that $v = \overline{v}_j + v_j$. The inductive step relies on

$$\mathcal{I}(\mathcal{P})v_j = f(\overline{u} + \overline{v}_j + v_j) - \mathcal{P}\overline{u} - \mathcal{P}\overline{v}_j - \mathcal{R}v_j \in \rho C^\infty(\overline{M}) + \mathcal{B}^{\mathrm{phg}}_{\delta_j+1}(M),$$

which in turn is a consequence of Remark 13. □

4. The free data

We now commence the proof of Theorem 1, and assume that (g, K) is a polyhomogeneous constant-mean-curvature asymptotically hyperbolic initial data set.

In this section we construct a family of free data $(\lambda_\varepsilon, \mu_\varepsilon)$, and subsequently establish several estimates for geometric quantities and differential operators associated to the family of metrics λ_ε. It is important that these estimates are uniform in $\varepsilon > 0$, in order that they lead to the convergence portion of Theorem 1. It is

our convention that, unless otherwise stated, all constants are independent of ε, provided ε is sufficiently small.

4.1. **Construction of the free data.** In order to construct a family of free data, we define a family of smooth cutoff functions. Let $\chi\colon \mathbb{R} \to [0,1]$ be a smooth, decreasing function such that

$$\chi(x) = 1 \text{ if } x \geq 2 \quad \text{and} \quad \chi(x) = 0 \text{ if } x \leq 1.$$

For $\varepsilon \in (0,1)$ define $\chi_\varepsilon\colon M \to [0,1]$ by $\chi_\varepsilon = \chi(\rho/\varepsilon)$. We note that $\operatorname{supp}\chi_\varepsilon \subset \{\rho > \varepsilon\}$ and that $\chi_\varepsilon = 1$ if $\rho \geq 2\varepsilon$. Furthermore, $d\chi_\varepsilon = \chi'(\rho/\varepsilon)\varepsilon^{-1}d\rho$ is supported in $\{\varepsilon \leq \rho \leq 2\varepsilon\}$. Thus, since $d\rho \in C^{k,\alpha}_1(M)$ for all $k \geq 1$ and $\alpha \in (0,1)$, we see that $\chi_\varepsilon \in C^{k,\alpha}(M)$, with bound independent of ε:

$$\|\chi_\varepsilon\|_{C^{k,\alpha}(M)} \leq C. \tag{4.1}$$

Let $\overline{b}$ be the smooth metric induced on ∂M by $\overline{g} = \rho^2 g$. We define a preferred background metric h by choosing $\overline{h}$ to be a smooth metric on $\overline{M}$ such that in a neighborhood of ∂M we have

$$\overline{h} = d\rho \otimes d\rho + \overline{b} \tag{4.2}$$

and setting $h = \rho^{-2}\overline{h}$. Let $\overline{\nabla}$ be the Levi-Civita connection associated to $\overline{h}$, and note that in the neighborhood of ∂M where (4.2) holds we have $\Delta_{\overline{h}}\rho = 0$.

We define, for sufficiently small $\varepsilon > 0$, the smooth metrics $\overline{\lambda}_\varepsilon$ on $\overline{M}$ by

$$\overline{\lambda}_\varepsilon := \chi_\varepsilon\, \overline{g} + (1 - \chi_\varepsilon)\overline{h}. \tag{4.3}$$

Setting $\lambda_\varepsilon = \rho^{-2}\overline{\lambda}_\varepsilon$, we define the family of free data $(\lambda_\varepsilon, \mu_\varepsilon)$ by

$$\lambda_\varepsilon := \rho^{-2}\overline{\lambda}_\varepsilon \quad \text{and} \quad \mu_\varepsilon := \chi_\varepsilon \Sigma = \chi_\varepsilon \rho^{-1}\overline{\Sigma}.$$

We emphasize that λ_ε are each a smoothly conformally compact asymptotically hyperbolic metric on M.

4.2. **Estimates for λ_ε.** We note the following properties of the metrics λ_ε.

Lemma 15. *Let $k \geq 0$ and $\alpha \in (0,1)$. We have*

$$\|g - \lambda_\varepsilon\|_{C^{k,\alpha}_1(M)} \leq C \quad \textit{and} \quad \|g - \lambda_\varepsilon\|_{C^{k,\alpha}(M)} \leq C\varepsilon.$$

This furthermore implies that for sufficiently small $\varepsilon > 0$ we have $\|g^{-1} - \lambda_\varepsilon^{-1}\|_{C^{k,\alpha}(M)} \leq C\varepsilon$.

Proof. Since $\overline{h}$ agrees with $\overline{g}$ at $\rho = 0$, we may apply Lemma 2 to conclude that $\overline{h} - \overline{g} \in C^{k,\alpha}_3(M)$ with bound independent of ε. Also recall (4.1), which shows that the functions $1 - \chi_\varepsilon$ are uniformly bounded in $C^{k,\alpha}(M)$. Our first claim now follows from the identity $\lambda_\varepsilon - g = \rho^{-2}(1 - \chi_\varepsilon)(\overline{h} - \overline{g})$.

Since the support of $\lambda_\varepsilon - g$ is in $\{\rho \le 2\varepsilon\}$, the first estimate implies the second. Finally, the estimate for the inverses comes from the second estimate applied to the series expansion $\lambda_\varepsilon^{-1} - g^{-1}$, centered at g. □

The following is immediate from the fact that $\overline{\lambda}_\varepsilon = \overline{h} = d\rho \otimes d\rho + \overline{b}$ in a collar neighborhood of the boundary.

Lemma 16. *We have $\mathcal{H}_{\overline{\lambda}_\varepsilon}(\rho) = 0$ and $|d\rho|^2_{\overline{\lambda}_\varepsilon} = 1$ along ∂M.*

We now obtain estimates on the scalar curvature of the metrics λ_ε. We first note that

$$R[\lambda_\varepsilon] + 6 = -6(|d\rho|^2_{\overline{\lambda}_\varepsilon} - 1) + 4\rho\Delta_{\overline{\lambda}_\varepsilon}\rho + \rho^2 R[\overline{\lambda}_\varepsilon]. \tag{4.4}$$

In a neighborhood of ∂M, where $\lambda_\varepsilon = \rho^{-2}\overline{h}$, we have

$$R[\lambda_\varepsilon] + 6 = \rho^2 R[\overline{h}] \in C^{k,\alpha}_2(M), \tag{4.5}$$

due to the fact that $|d\rho|^2_{\overline{h}} \equiv 1$ and $\Delta_{\overline{h}}\rho \equiv 0$ near ∂M. However, we do not have a uniform estimate on $R[\lambda_\varepsilon] + 6$ in $C^{k,\alpha}_2(M)$. Rather, we obtain the following.

Proposition 17. *Let $k \ge 0$ and $\alpha \in (0,1)$. For sufficiently small $\varepsilon > 0$ we have*

$$\|R[\lambda_\varepsilon] - R[g]\|_{C^{k,\alpha}_1(M)} \le C \quad \text{and} \quad \|R[\lambda_\varepsilon] - R[g]\|_{C^{k,\alpha}(M)} \le C\varepsilon.$$

Proof. We make use of the formula (4.4), analyzing each term on the right side. The scalar curvature $R[\overline{\lambda}_\varepsilon]$ is the sum of contractions of terms of the form

$$(\overline{\lambda}_\varepsilon)^{-1} \otimes (\overline{\lambda}_\varepsilon)^{-1} \otimes (\overline{\lambda}_\varepsilon)^{-1} \otimes \overline{\nabla}\,\overline{\lambda}_\varepsilon \otimes \overline{\nabla}\,\overline{\lambda}_\varepsilon \quad \text{and} \quad (\overline{\lambda}_\varepsilon)^{-1} \otimes (\overline{\lambda}_\varepsilon)^{-1} \otimes \overline{\nabla}^2\overline{\lambda}_\varepsilon;$$

The scalar curvature of $\overline{g}$ is comprised of analogous terms. From Lemma 15 we have

$$\|(\overline{\lambda}_\varepsilon)^{-1} - (\overline{g})^{-1}\|_{C^{k,\alpha}_{-2}(M)} \le \|\lambda_\varepsilon^{-1} - g^{-1}\|_{C^{k,\alpha}(M)} \le C.$$

Likewise, both $\|\overline{\nabla}(\overline{\lambda}_\varepsilon - \overline{g})\|_{C^{k,\alpha}_3(M)}$ and $\|\overline{\nabla}^2(\overline{\lambda}_\varepsilon - \overline{g})\|_{C^{k,\alpha}_3(M)}$ can be bounded by

$$\|\overline{\lambda}_\varepsilon - \overline{g}\|_{C^{k+2,\alpha}_3(M)} \le \|\lambda_\varepsilon - g\|_{C^{k+2,\alpha}_1(M)} \le C.$$

We now conclude that

$$\|\rho^2(R[\overline{\lambda}_\varepsilon] - R[\overline{g}])\|_{C^{k,\alpha}_1(M)} \le C.$$

Similar reasoning, using that $d\rho \in C^{k,\alpha}_1(M)$ and $\overline{\nabla} d\rho \in C^{k,\alpha}_2(M)$, yields

$$\|\rho(\Delta_{\overline{\lambda}_\varepsilon}\rho - \Delta_{\overline{g}}\rho)\|_{C^{k,\alpha}_1(M)} \le C.$$

Finally, we estimate the function

$$\eta = (|d\rho|^2_{\overline{\lambda}_\varepsilon} - 1) - (|d\rho|^2_{\overline{g}} - 1) = ((\overline{\lambda}_\varepsilon)^{-1} - (\overline{g})^{-1})(d\rho, d\rho).$$

Lemma 15 implies that $\|\eta\|_{C^{2,\alpha}(M)}$ and $\|\overline{\nabla}\eta\|_{C_1^{1,\alpha}(M)}$ are uniformly bounded in ε. Since η vanishes at $\rho = 0$ we may apply Lemma 2 to conclude that $\|\eta\|_{C_1^{2,\alpha}(M)}$ is uniformly bounded in ε. This establishes the first estimate in the lemma. The second estimate follows from the first due to the fact that λ_ε agrees with g for $\rho \geq 2\varepsilon$. □

4.3. **Estimates for geometric operators defined by λ_ε.** Here we record several consequences of Proposition 3 and Lemma 15.

Proposition 18. *For any $k \geq 0$ and $\alpha \in [0,1)$, and for any $\delta \in \mathbb{R}$, there is a constant $C > 0$, independent of sufficiently small ε, such that the following hold:*

(a) For any tensor field $u \in C_\delta^{k+1,\alpha}(M)$ we have

$$\| \operatorname{div}_{\lambda_\varepsilon} u\|_{C_\delta^{k,\alpha}(M)} \leq C\|u\|_{C_\delta^{k+1,\alpha}(M)}.$$

(b) For any vector field $X \in C_\delta^{k+1,\alpha}(M)$ we have

$$\|\mathcal{D}_{\lambda_\varepsilon} X\|_{C_\delta^{k,\alpha}(M)} \leq C\|X\|_{C_\delta^{k+1,\alpha}(M)}.$$

Proof. For the first claim, we note that the estimate holds with λ_ε replaced by g. Since

$$\| \operatorname{div}_{\lambda_\varepsilon} u\|_{C_\delta^{k,\alpha}(M)} \leq \| \operatorname{div}_{\lambda_\varepsilon} u - \operatorname{div}_g u\|_{C_\delta^{k,\alpha}(M)} + \| \operatorname{div}_g u\|_{C_\delta^{k,\alpha}(M)}$$

we may invoke Proposition 3 and Lemma 15 to obtain the desired estimate. The proof of the second claim follows from analogous reasoning. □

Due to Proposition 4 the vector Laplacian $L_{\lambda_\varepsilon} : C_\delta^{k+2,\alpha}(M) \to C_\delta^{k,\alpha}(M)$ is invertible for each $\varepsilon > 0$, $k \geq 0$, $\alpha \in (0,1)$ and $\delta \in (-1,3)$. In particular, there exist constants C_ε, depending on ε, such that $\|X\|_{C_\delta^{k+2,\alpha}(M)} \leq C_\varepsilon \|L_{\lambda_\varepsilon} X\|_{C_\delta^{k,\alpha}(M)}$. The linearized Licherowicz operator that appears in §5 is similarly invertible for each λ_ε. We now show that the invertibility estimates can be made uniform in ε.

Proposition 19. *Let $k \geq 0$, $\alpha \in (0,1)$, and $\delta \in (-1,3)$. Furthermore, let the functions $\kappa, \kappa_\varepsilon \in C_1^{k,\alpha}(M)$ be such that $\|\kappa_\varepsilon - \kappa\|_{C^{k,\alpha}(M)} \leq C\varepsilon$ and $3 + \kappa \geq 0$. Then there exists a constant $C > 0$ such that:*

(a) for all vector fields $X \in C_\delta^{k+2,\alpha}(M)$ and for all sufficiently small $\varepsilon > 0$ we have

$$\|X\|_{C_\delta^{k+2,\alpha}(M)} \leq C\|L_{\lambda_\varepsilon} X\|_{C_\delta^{k,\alpha}(M)},$$

and

(b) for all functions $u \in C_\delta^{k+2,\alpha}(M)$ and for all sufficiently small $\varepsilon > 0$ we have

$$\|u\|_{C_\delta^{k+2,\alpha}(M)} \leq C\|\Delta_{\lambda_\varepsilon} u - (3 + \kappa_\varepsilon)u\|_{C_\delta^{k,\alpha}(M)}.$$

Proof. From Proposition 4(a) we have

$$\begin{aligned} \|X\|_{C^{k+2,\alpha}_\delta(M)} &\le C\|L_g X\|_{C^{k,\alpha}_\delta(M)} \\ &\le C\left(\|L_g X - L_{\lambda_\varepsilon}X\|_{C^{k,\alpha}_\delta(M)} + \|L_{\lambda_\varepsilon}X\|_{C^{k,\alpha}_\delta(M)}\right). \end{aligned} \tag{4.6}$$

From Proposition 3 we have

$$\|L_g X - L_{\lambda_\varepsilon}X\|_{C^{k,\alpha}_\delta(M)} \le C\|g-\lambda_\varepsilon\|_{C^{k+2,\alpha}(M)}\|X\|_{C^{k+2,\alpha}_\delta(M)}.$$

Making use of Lemma 15, we see that this term may be absorbed in to the left side of (4.6) when $\varepsilon > 0$ is small; this proves the first invertibility estimate. The second estimate follows from a similar argument applied to Proposition 4(b); the details are left to the reader. □

4.4. **Estimates for μ_ε.**

Lemma 20. *Let $k \ge 0$ and $\alpha \in (0,1)$. There exists a constant $C > 0$ such that*

$$\|\mu_\varepsilon - \Sigma\|_{C^{k,\alpha}_1(M)} \le C, \qquad \|\mu_\varepsilon - \Sigma\|_{C^{k,\alpha}(M)} \le C\varepsilon,$$
$$\|\operatorname{div}_{\lambda_\varepsilon}\mu_\varepsilon\|_{C^{k,\alpha}_1(M)} \le C, \qquad \|\operatorname{div}_{\lambda_\varepsilon}\mu_\varepsilon\|_{C^{k,\alpha}(M)} \le C\varepsilon.$$

Furthermore, $\operatorname{div}_{\lambda_\varepsilon}\mu_\varepsilon \in C^{k,\alpha}_\delta(M)$ *for all* $\delta \in \mathbb{R}$.

Proof. First recall (4.1) and note that $\overline{\Sigma} \in C^{1,\alpha}_2(M)$; thus $\mu_\varepsilon = \rho^{-1}\chi_\varepsilon\overline{\Sigma}$ is uniformly bounded in $C^{1,\alpha}_1(M)$, which implies the first estimate. The second estimate follows from this and the fact that the support of $\mu_\varepsilon - \Sigma$ is contained in the region where $\rho \le 2\varepsilon$.

The uniform bound on μ_ε in $C^{1,\alpha}_1(M)$, together with Proposition 18(a), implies that $\operatorname{div}_{\lambda_\varepsilon}\mu_\varepsilon$ is uniformly bounded in $C^{0,\alpha}_1(M)$. Since λ_ε agrees with g and μ_ε agrees with $\rho^{-1}\overline{\Sigma}$ for $\rho \ge 2\varepsilon$, we see from (1.3) that $\operatorname{div}_{\lambda_\varepsilon}\mu_\varepsilon$ is supported in the region $\varepsilon \le \rho \le 2\varepsilon$. This, together with the third estimate, yields the fourth estimate.

Finally, the fact that μ_ε is compactly supported implies that $\operatorname{div}_{\lambda_\varepsilon}\mu_\varepsilon \in C^{0,\alpha}_\delta(M)$ for all δ. □

5. Construction of approximating initial data

5.1. **Analysis of the conformal momentum constraint.** For each free data set $(\lambda_\varepsilon, \mu_\varepsilon)$, Propositions 4 and 9 guarantee that there exists a unique $W_\varepsilon \in \rho^3 C^0_{\mathrm{phg}}(\overline{M})$ such that

$$L_{\lambda_\varepsilon}W_\varepsilon = -\operatorname{div}_{\lambda_\varepsilon}\mu_\varepsilon \tag{5.1}$$

and $\mathcal{D}_{\lambda_\varepsilon}W_\varepsilon \in C^0_{\mathrm{phg}}(\overline{M})$. By Proposition 19 there is a constant C such that for all sufficiently small $\varepsilon > 0$ we have

$$\|W_\varepsilon\|_{C^{k+2,\alpha}_\delta(M)} \le C\|\operatorname{div}_{\lambda_\varepsilon}\mu_\varepsilon\|_{C^{k,\alpha}_\delta(M)}$$

for $\delta = 0, 1$. The estimates for μ_ε in Lemma 20 now imply the following estimates for the solutions W_ε to (5.1).

Lemma 21. *Let $k \geq 0$ and $\alpha \in (0,1)$. There exists a constant $C > 0$ such that*

$$\|W_\varepsilon\|_{C_1^{k,\alpha}(M)} \leq C, \qquad \|W_\varepsilon\|_{C^{k,\alpha}(M)} \leq C\varepsilon,$$
$$\|\mathcal{D}_{\lambda_\varepsilon} W_\varepsilon\|_{C_1^{k,\alpha}(M)} \leq C, \qquad \|\mathcal{D}_{\lambda_\varepsilon} W_\varepsilon\|_{C^{k,\alpha}(M)} \leq C\varepsilon.$$

We define the tensors σ_ε by

$$\sigma_\varepsilon := \mu_\varepsilon + \mathcal{D}_{\lambda_\varepsilon} W_\varepsilon$$

and record the following consequence of Lemmas 15, 20, and 21.

Lemma 22. *Let $k \geq 0$, $\alpha \in (0,1)$ and let $\varepsilon > 0$ be sufficiently small. The function $|\sigma_\varepsilon|^2_{\lambda_\varepsilon}$ is in $C_2^{k,\alpha}(M)$, and satisfies*

$$\||\sigma_\varepsilon|^2_{\lambda_\varepsilon} - |\Sigma|^2_{\lambda_\varepsilon}\|_{C_2^{k,\alpha}(M)} \leq C \quad \textit{and} \quad \||\sigma_\varepsilon|^2_{\lambda_\varepsilon} - |\Sigma|^2_{\lambda_\varepsilon}\|_{C_1^{k,\alpha}(M)} \leq C\varepsilon.$$

Finally, we address smoothness of the tensor fields σ_ε. The strategy is to employ Proposition 12 of §3.

Proposition 23. *The solution W_ε of* (5.1) *and the tensor field σ_ε extend smoothly to $\overline{M}$.*

Proof. A direct computation shows that the indicial map $I_s(L_{\lambda_\varepsilon})$ is

$$I_s(L_{\lambda_\varepsilon})Y = -\frac{1}{2}\left(Y + \frac{1}{3}Y(\rho)\,\mathrm{grad}_{\overline{\lambda}_\varepsilon}\,\rho\right)(s^2 - 4s).$$

Thus L_{λ_ε} satisfies Assumption L with the highest characteristic exponent of $\mu = 4$. By Proposition 9 we have that W_ε is polyhomogeneous, while Proposition 4 implies $W_\varepsilon \in C^k_\delta(M)$ for all $k \geq 0$ and $\delta < 3$. Since $L_{\lambda_\varepsilon} W_\varepsilon = -\operatorname{div}_{\lambda_\varepsilon} \mu_\varepsilon$ extends smoothly to $\overline{M}$ and since by Lemma 20 $\operatorname{div}_{\lambda_\varepsilon} \mu_\varepsilon \in C^k_\delta(M)$ for all $k \geq 0$ and all $\delta \in \mathbb{R}$, we are in position to apply Proposition 12. Consequently, W_ε extends smoothly to $\overline{M}$, and thus σ_ε does as well. □

5.2. **Analysis of the Lichnerowicz equation.** From Proposition 9(b) there exists, for each sufficiently small $\varepsilon > 0$, a unique positive polyhomogeneous function $\phi_\varepsilon \in C^2_{\mathrm{phg}}(\overline{M})$ such that

$$0 = \mathcal{N}_\varepsilon(\phi_\varepsilon) := \Delta_{\lambda_\varepsilon}\phi_\varepsilon - \frac{1}{8}R[\lambda_\varepsilon]\phi_\varepsilon + \frac{1}{8}|\sigma_\varepsilon|^2_{\lambda_\varepsilon}\phi_\varepsilon^{-7} - \frac{3}{4}\phi_\varepsilon^5, \quad \phi_\varepsilon|_{\partial M} = 1. \tag{5.2}$$

In order to obtain estimates on $\phi_\varepsilon - 1$ we first show that the constant function $\phi = 1$ is an approximate solution of (5.2).

Lemma 24. *Let $k \geq 0$ and $\alpha \in (0,1)$. For each sufficiently small $\varepsilon > 0$ we have $\mathcal{N}_\varepsilon(1) \in C_1^{0,\alpha}(M)$ with*

$$\|\mathcal{N}_\varepsilon(1)\|_{C_1^{k,\alpha}(M)} \leq C \quad \text{and} \quad \|\mathcal{N}_\varepsilon(1)\|_{C^{k,\alpha}(M)} \leq C\varepsilon. \tag{5.3}$$

Proof. Using (1.3) we have

$$\mathcal{N}_\varepsilon(1) = -\frac{1}{8}\left(R[\lambda_\varepsilon] - R[g]\right) + \frac{1}{8}\left(|\sigma_\varepsilon|^2_{\lambda_\varepsilon} - |\Sigma|^2_{\lambda_\varepsilon}\right) + \frac{1}{8}\left(|\Sigma|^2_{\lambda_\varepsilon} - |\Sigma|^2_g\right).$$

Estimates (5.3) are now immediate from Proposition 17, Lemma 22, and Lemma 15. □

The linearization of $\mathcal{N}_\varepsilon$ at $\phi = 1$ is the operator

$$\mathcal{L}_\varepsilon := \Delta_{\lambda_\varepsilon} - (3 + \kappa_\varepsilon), \tag{5.4}$$

where

$$\kappa_\varepsilon = \frac{1}{8}\left(R[\lambda_\varepsilon] + 6\right) + \frac{7}{8}|\sigma_\varepsilon|^2_{\lambda_\varepsilon}.$$

We now prove the properties of κ_ε needed in order to apply Proposition 19. To that end we set $\kappa = |\Sigma|^2_g$ and note that $\kappa \in C_2^{k,\alpha}(M)$ for all $k \geq 0$ and $\alpha \in (0,1)$.

Lemma 25. *For all $k \geq 0$ and $\alpha \in (0,1)$ and sufficiently small $\varepsilon > 0$ we have:*

(a) $\kappa_\varepsilon \in C_1^{k,\alpha}(M)$.
(b) $\|\kappa_\varepsilon - \kappa\|_{C^{k,\alpha}(M)} \leq C\varepsilon$.

Proof. The fact that $\kappa_\varepsilon \in C_1^{k,\alpha}(M)$ is immediate from Proposition 17 and Lemma 22. Using (1.3) we can express $\kappa_\varepsilon - \kappa$ as

$$\kappa_\varepsilon - \kappa = \frac{1}{8}\left(R[\lambda_\varepsilon] - R[g]\right) + \frac{7}{8}\left(|\sigma_\varepsilon|^2_{\lambda_\varepsilon} - |\Sigma|^2_g\right).$$

The $C^{k,\alpha}(M)$ estimate on the difference of scalar curvatures follows from Proposition 17, while the remaining estimate follows from Lemmas 15 and 22. □

We now see from Proposition 19 that for all $\delta \in (-1,3)$ and sufficiently small $\varepsilon > 0$ the mapping

$$\mathcal{L}_\varepsilon \colon C_\delta^{k+2,\alpha}(M) \to C_\delta^{k,\alpha}(M)$$

defined in (5.4) is an isomorphism with inverse bounded uniformly in ε.

We now proceed to obtain estimates for ϕ_ε by viewing $\phi_\varepsilon - 1$ as a fixed point of the map

$$\mathcal{G}_\varepsilon \colon u \to -\mathcal{L}_\varepsilon^{-1}\left(\mathcal{N}_\varepsilon(1) + \mathcal{Q}_\varepsilon(u)\right),$$

where

$$\begin{aligned}\mathcal{Q}_\varepsilon(u) &:= \mathcal{N}_\varepsilon(1+u) - \mathcal{N}_\varepsilon(1) - \mathcal{L}_\varepsilon(u)\\ &= \frac{1}{8}|\sigma_\varepsilon|^2_{\lambda_\varepsilon}\left((1+u)^{-7} - 1 + 7u\right) - \frac{3}{4}\left((1+u)^5 - 1 - 5u\right).\end{aligned}$$

In preparation, we define for each $r_0, r_1 > 0$, which are not necessarily independent of ε and a fixed integer $k \in \mathbb{N}_0$, the collection of functions

$$X(r_0, r_1) = \{u \in C_1^{k+2,\alpha}(M) \mid \|u\|_{C_1^{k+2,\alpha}(M)} \leq r_1 \text{ and } \|u\|_{C^{k+2,\alpha}(M)} \leq r_0\}.$$

Note that for all $r_0, r_1 > 0$, the set $X(r_0, r_1)$ is a complete metric space with respect to the norm $|||u|||_X := \|u\|_{C_1^{k+2,\alpha}(M)} + \|u\|_{C^{k+2,\alpha}(M)}$.

We require the following mapping properties of $\mathcal{Q}_\varepsilon$.

Lemma 26. *Let $k \geq 0$ and $\alpha \in (0,1)$. There exists $r_* > 0$ and continuous function $F\colon [0, r_*] \times [0, r_*] \to [0, \infty)$, both independent of $\varepsilon > 0$, such that $F(0,0) = 0$ and such that for each $\delta \in [0,1]$ we have*

$$\|\mathcal{Q}_\varepsilon(u) - \mathcal{Q}_\varepsilon(v)\|_{C_\delta^{k,\alpha}(M)} \leq F\left(\|u\|_{C^{k,\alpha}(M)}, \|v\|_{C^{k,\alpha}(M)}\right) \|u - v\|_{C_\delta^{k,\alpha}(M)}$$

for all $u, v \in X(r_0, r_1)$ with $r_0 \in [0, r_*]$ and $r_1 > 0$. In particular

$$\|\mathcal{Q}_\varepsilon(u)\|_{C_\delta^{k,\alpha}(M)} \leq F(r_0, 0) \|u\|_{C_\delta^{k,\alpha}(M)}.$$

Proof. Set

$$H_l(u, v) = u^{l-1}v + u^{l-2}v^2 + \cdots + uv^{l-1}.$$

With

$$Q_1(u) := u^5 \quad \text{and} \quad Q_2(u) := u^{-7}$$

we have

$$Q_1(u) - Q_1(v) = (u - v) \sum_{l=2}^{5} \binom{5}{l} H_l(u, v)$$

and

$$Q_2(u) - Q_2(v) = (u - v) \sum_{l=2}^{\infty} (-1)^l \binom{l+6}{l} H_l(u, v)$$

provided $|u|$ and $|v|$ are less than 1. The uniform bound on $|\sigma_\varepsilon|^2_{\lambda_\varepsilon}$ provided by Lemma 22 implies that there are constants C_* and C_l, $l \in \mathbb{N}$, such that

$$F(u, v) = C_* \sum_{l=2}^{\infty} C_l H_l(u, v)$$

converges uniformly and has the desired properties. □

We now obtain the desired contraction property of $\mathcal{G}_\varepsilon$.

Lemma 27. *There exists constants $\varepsilon_* > 0$ and $C_* > 0$ such that for each $\varepsilon \in (0, \varepsilon_*]$ the map $\mathcal{G}_\varepsilon$ is a contraction mapping $X(C_*\varepsilon, C_*) \to X(C_*\varepsilon, C_*)$.*

Proof. Let $u \in X(r_0, r_1)$ and $\delta \in \{0, 1\}$. By the uniform invertibility of $\mathcal{L}_\varepsilon^{-1}$ (cf. Proposition 19 and Lemma 25) we have

$$\begin{aligned}\|\mathcal{G}_\varepsilon(u)\|_{C_\delta^{k+2,\alpha}(M)} &= \|\mathcal{L}_\varepsilon^{-1}\left(\mathcal{N}_\varepsilon(1) + \mathcal{Q}_\varepsilon(u)\right)\|_{C_\delta^{k+2,\alpha}(M)} \\ &\leq C\|\mathcal{N}_\varepsilon(1)\|_{C_\delta^{k,\alpha}(M)} + C\|\mathcal{Q}_\varepsilon(u)\|_{C_\delta^{k,\alpha}(M)}.\end{aligned}$$

Using Lemma 24 and Lemma 26 we obtain

$$\|\mathcal{G}_\varepsilon(u)\|_{C^{k+2,\alpha}(M)} \le C'\varepsilon + C'' F(r_0, 0)\|u\|_{C^{k+2,\alpha}(M)}$$
$$\|\mathcal{G}_\varepsilon(u)\|_{C_1^{k+2,\alpha}(M)} \le C' + C'' F(r_0, 0)\|u\|_{C_1^{k+2,\alpha}(M)}$$

for some constants $C', C'' > 0$ independent of ε. Choosing $C_* > 2C'$ and, using the fact that $F(0,0) = 0$, choosing ε_* small enough that $C''F(C_*\varepsilon_*, 0) < 1/2$ ensures that

$$\mathcal{G}_\varepsilon \colon X(C_*\varepsilon, C_*) \to X(C_*\varepsilon, C_*)$$

whenever $\varepsilon \in (0, \varepsilon_*]$.

We furthermore have

$$\|\mathcal{G}_\varepsilon(u) - \mathcal{G}_\varepsilon(v)\|_{C_\delta^{k+2,\alpha}(M)} \le C\|\mathcal{Q}_\varepsilon(u) - \mathcal{Q}_\varepsilon(v)\|_{C_\delta^{k,\alpha}(M)}.$$

Since $F(0,0) = 0$, Lemma 26 implies that we can choose $\varepsilon_* > 0$ such that $\mathcal{G}_\varepsilon$ is a contraction for $\varepsilon \in (0, \varepsilon_*]$. □

The contraction property of $\mathcal{G}_\varepsilon$, together with the Banach fixed point theorem, immediately leads to following.

Proposition 28. *Let $k \ge 0$ and $\alpha \in (0,1)$. There exists $\varepsilon_* > 0$ and constant $C > 0$ such that whenever $\varepsilon \in (0, \varepsilon_*)$ we have $\phi_\varepsilon - 1 \in C_1^{k,\alpha}(M)$ and*

$$\|\phi_\varepsilon - 1\|_{C^{k,\alpha}(M)} \le C\varepsilon.$$

We now analyze the regularity on $\overline{M}$ of solutions ϕ_ε of (5.2). We do so by writing (5.2) in terms of the auxiliary variable

$$u = \phi_\varepsilon - u_0, \quad \text{where} \quad u_0 = 1 - \frac{1}{24}\rho^2 \operatorname{R}[\overline{h}].$$

This particular change of variable is motivated by the fact that, while Lemma 24 shows that the function 1 is an approximate solution to (5.2), the function u_0 constitutes a better approximate solution. We make this precise in the following lemma.

Lemma 29. *Let $k \ge 0$ and $\alpha \in (0,1)$. For each sufficiently small $\varepsilon > 0$ we have $\mathcal{N}_\varepsilon(u_0) \in \rho^4 C^\infty(\overline{M})$.*

Proof. It suffices to perform the computation in the collar neighborhood of the boundary where $\lambda_\varepsilon = h$. There we have $\operatorname{R}[h] = -6 + \rho^2 \operatorname{R}[\overline{h}]$ and $\langle d\rho, d\operatorname{R}[\overline{h}]\rangle_{\overline{h}} = 0$; the latter can be seen as a consequence of the fact that $\operatorname{grad}_{\overline{h}}\rho$ is a Killing vector field in the collar neighborhood of ∂M. A direct computation now shows that

$$\Delta_h u_0 = \rho^2 \Delta_{\overline{h}} u_0 - \rho\langle d\rho, du_0\rangle_{\overline{h}} = \rho^4 \Delta_{\overline{h}} \operatorname{R}[\overline{h}] \in \rho^4 C^\infty(\overline{M}).$$

On the other hand, Propositions 9 and 23 imply $|\sigma_\varepsilon|^2_{\lambda_\varepsilon} \in \rho^4 C^\infty(\overline{M})$. Using this fact we obtain

$$\begin{aligned}
&\frac{1}{8}\,\mathrm{R}[h]u_0 - \frac{1}{8}|\sigma_\varepsilon|^2_h u_0^{-7} + \frac{3}{4}u_0^5 \\
&\qquad = \frac{1}{8}\left(-6 + \frac{5}{4}\rho^2\,\mathrm{R}[\overline{h}]\right) + \frac{3}{4}\left(1 - \frac{5}{24}\rho^2\,\mathrm{R}[\overline{h}]\right) + \rho^4 C^\infty(\overline{M}) \in \rho^4 C^\infty(\overline{M}).
\end{aligned}$$

This completes the proof. □

Proposition 30. *The solution ϕ_ε of* (5.2) *extends to a smooth function on $\overline{M}$.*

Proof. We see from Lemma 29 that

$$\Delta_h u = \frac{1}{8}\,\mathrm{R}[h]u - \frac{1}{8}|\sigma_\varepsilon|^2_h\left((u_0+u)^{-7} - u_0^{-7}\right) + \frac{3}{4}\left((u_0+u)^5 - u_0^5\right) + \rho^4 C^\infty(\overline{M}).$$

Since $\frac{1}{8}\,\mathrm{R}[h] = -\frac{3}{4} + \rho^2 C^\infty(\overline{M})$, $|\sigma_\varepsilon|^2_{\lambda_\varepsilon} \in \rho^4 C^\infty(\overline{M})$, and $\frac{15}{4}u_0^4 = \frac{15}{4} + \rho^2 C^\infty(\overline{M})$, we have

$$\Delta_{\lambda_\varepsilon} u - 3u = f(u), \tag{5.5}$$

where near $\rho = 0$ the function $f(u)$ has the uniformly and absolutely convergent power series

$$f(u) = \sum_{l=0}^{\infty} a_l u^l$$

with $a_0 \in \rho^4 C^\infty(\overline{M})$, $a_1 \in \rho^2 C^\infty(\overline{M})$, and $a_l \in C^\infty(\overline{M})$ for $l \geq 2$. In particular, f satisfies Assumption F. Also note that, by Proposition 9(b), $u \in \rho^2 C^0_{\mathrm{phg}}(\overline{M})$; consequently $f(u) \in C^{k,\alpha}_4$ for all $k \geq 0$ and $\alpha \in (0,1)$. Applying Proposition 4 now yields $u \in C^k_\delta(M)$ for all $k \geq 0$ and $\delta < 3$.

Finally, we observe that the indicial map of $\Delta_{\lambda_\varepsilon} - 3$ is

$$I_s(\Delta_{\lambda_\varepsilon} - 3) = (s-3)(s+1).$$

In particular, $\Delta_{\lambda_\varepsilon} - 3$ satisfies Assumption L with the highest characteristic exponent of $\mu = 3$. Invoking Proposition 14 we conclude that u and ϕ_ε extend to functions in $C^\infty(\overline{M})$. □

5.3. **The proof of Theorem 1.** We now construct the approximating initial data and show that they satisfy the shear-free condition, are smoothly conformally compact, and have the desired convergence property.

The solutions W_ε to (5.1) and ϕ_ε to (5.2) give rise to initial data sets $(g_\varepsilon, K_\varepsilon)$ determined by

$$\begin{aligned}
g_\varepsilon &= \phi_\varepsilon^4 \lambda_\varepsilon \\
K_\varepsilon &= \Sigma_\varepsilon - g_\varepsilon = \phi_\varepsilon^{-2}\sigma_\varepsilon - \phi_\varepsilon^4 \lambda_\varepsilon.
\end{aligned} \tag{5.6}$$

By Propositions 23 and 30 we see that $\overline{g}_\varepsilon = \rho^2 g_\varepsilon$ and $\overline{\Sigma}_\varepsilon = \rho(K_\varepsilon + g_\varepsilon)$ extend smoothly to $\overline{M}$.

To see that $(g_\varepsilon, K_\varepsilon)$ is shear-free note that Lemma 16, Proposition 5, and the fact that $\phi_\varepsilon = 1$ along ∂M imply

$$\mathcal{H}_{\overline{g}_\varepsilon}(\rho)\big|_{\partial M} = 0.$$

In addition, we have

$$\overline{\Sigma}_\varepsilon\big|_{\partial M} = \rho\sigma_\varepsilon|_{\partial M} = \rho\left(\mu_\varepsilon + \mathcal{D}_{\lambda_\varepsilon} W_\varepsilon\right)|_{\partial M}.$$

By definition, μ_ε vanishes along ∂M. Furthermore, Proposition 9 implies that $\mathcal{D}_{\lambda_\varepsilon} W_\varepsilon \in C^0_{\mathrm{phg}}(\overline{M})$, and thus we see that $\overline{\Sigma}_\varepsilon$ vanishes along ∂M. Consequently, the approximating family of initial data $(g_\varepsilon, K_\varepsilon)$ satisfies the shear-free condition.

Finally, we prove the following convergence property.

Proposition 31. *Let $k \geq 0$ and $\alpha \in (0,1)$. Then*

$$\|g_\varepsilon - g\|_{C^{k,\alpha}(M)} \leq C\varepsilon, \quad \|K_\varepsilon - K\|_{C^{k,\alpha}(M)} \leq C\varepsilon$$

for some constant C independent of ε.

Proof. We have

$$g_\varepsilon - g = \phi_\varepsilon^4(\lambda_\varepsilon - g) + (\phi_\varepsilon^4 - 1)g.$$

From Lemma 15 we see that the $C^{k,\alpha}$ norm of the first term is $\mathcal{O}(\varepsilon)$, while the second term can be estimated using Proposition 28.

Note that $K - K_\varepsilon = \Sigma - \Sigma_\varepsilon - (g - g_\varepsilon)$. Thus it suffices to estimate

$$\Sigma - \Sigma_\varepsilon = \Sigma - \phi_\varepsilon^{-2}(\mu_\varepsilon + \mathcal{D}_{\lambda_\varepsilon} W_\varepsilon).$$

Due to Lemma 21 and Proposition 28, it suffices to estimate $\Sigma - \mu_\varepsilon$. This, however, is accomplished in Lemma 20. □

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